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Probability distributions for second-order processes driven by Gaussian noise

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We derive a Fokker-Planck equation for the joint probability density of the displacement and the velocity of a free particle subjected to an exponentially correlated Gaussian force. This equation is solved analytically in the limits $t \ll \tau$, $t \gg \tau$ and for $\tau=0$ (white noise), where τ is the correlation time. The parameters (moments) which determine the joint density are calculated including terms up to order t^2/τ^2 for $t \ll \tau$, and up to order τ/t for $t \gg \tau$. For $t \ll \tau$ the marginal distribution of displacements is exactly Gaussian, to the considered order. A Gaussian distribution derived approximately for $t \gg \tau$ is suggested to be exact, on the basis of independent, exact calculations of low-order moments. For Gaussian white noise, the joint density is obtained exactly and yields a Gaussian distribution of displacements, with the familiar superdiffusive form for the mean-square deviation. The marginal distribution of velocity obeys an exact diffusion equation with a variable diffusion coefficient, for arbitrary τ.

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I. INTRODUCTION

The instantaneous displacement $x(t)$ of a free particle of mass m subjected to a purely random, uniform force is the simplest example of a second-order process, which is governed by the second-order stochastic differential equation

$$\ddot{x}(t)=\frac{1}{m}f(t) . \tag{1}$$

This process is clearly equivalent to two coupled first-order processes corresponding to $x(t)$ and to the instantaneous velocity $v(t)$, since

$$\dot{x}(t)=v(t) , \tag{2a}$$

$$\dot{v}(t)=\frac{1}{m}f(t) . \tag{2b}$$

The equivalent form [Eqs. (2a) and (2b)] for the second-order process (1) is convenient for deriving a differential equation for the joint probability density $p(x,v,t)$ of a displacement x and a velocity v for the particle. For a Gaussian random force this equation takes the form of a two-dimensional Fokker-Planck equation whose coefficients are generally time dependent. Despite their fundamental interest, explicit solutions for distributions of second- and higher-order processes occurring in physics, chemistry, or engineering (e.g., control, filtering communications) are rare. In particular, for the free particle in a random field $f(t)$ one is interested in the joint density $p(x,v,t)$, as well as in the marginal distributions of displacements and velocities:

$$p(x,t)=\int_{-\infty}^{\infty} dv\, p(x,v,t) , \tag{3a}$$

$$p(v,t)=\int_{-\infty}^{\infty} dx\, p(x,v,t) , \tag{3b}$$

which govern the probabilities of the independently defined processes (1) and (2b), respectively.

Recently, however, Masoliver [1] was able to obtain exact differential equations for $p(x,v,t)$, $p(x,t)$, and $p(v,t)$ for dichotomous noise $f(t)$, and to study solutions for $p(x,t)$ and $p(v,t)$ for large t. The random force alternately takes values $\pm a$ with a distribution $\psi(t)$ of time intervals between changes of sign. Masoliver's detailed solutions apply in the case where $\psi(t)$ is an exponential [2]. This corresponds to dichotomous Markovian noise, where the autocorrelation function of $f(t)$ depends exponentially on the time difference:

$$\begin{aligned} &\langle f(t)f(t')\rangle=f_0^2h(t-t') ,\\ &\langle f(t)\rangle=0 ,\\ &h(t-t')=\frac{1}{2\tau}\exp\left[-\frac{|t-t'|}{\tau}\right] , \end{aligned} \tag{4}$$

with a correlation time τ equal to half the average time between sign changes. Masoliver's work was motivated, in particular, by recent studies of Pawula [3] of the probability density for the output of second-order oscillating filters driven by dichotomous (or random telegraph) noise.

The purpose of the present paper is to study the above distributions in the simple case where $f(t)$ is a Gaussian random force [4] with an exponential correlation of the form (4). This type of random force is known as Ornstein-Uhlenbeck noise, which further reduces to Gaussian white noise,

$$\langle f(t)f(t')\rangle=f_0^2\delta(t-t'),\quad \langle f(t)\rangle=0 , \tag{5}$$

in the limit $\tau=0$ [since $\lim_{\tau\to 0}h(t)=\delta(t)$]. We also recall that an exponential autocorrelation of the form (4) is a general feature of Markovian noise (Doob's theorem). Equation (1) with a Gaussian force $f(t)$ corresponds to a Langevin model without dissipation, which is currently being used, e.g., to model reaction kinetics [5] in systems involving mixing of low-viscosity liquids, or in systems that display turbulence or which are undergoing a fast increase in temperature. As is well known, the random motion described by Eq. (1) with a force correlation (4) is qualitatively different from the corresponding Brownian motion described by the Langevin equation: one finds, for example [6], that the mean-squared displacement increases superdiffusively, i.e., $\langle x^2(t)\rangle \sim t^3$, for $t \gg \tau$, while increasing as t^4 for $t \ll \tau$. The latter behavior is due to the fact that for $t \ll \tau$ the Gaussian force is effectively constant, which leads to a constant random acceleration of the particle.

On the other hand, the influence of Ornstein-Uhlenbeck noise on the dynamical behavior of nonlinear first-order processes [i.e., processes such as (2b), with an additional nonrandom force on the right-hand side (rhs)], has been studied extensively in recent years [7], mostly for small correlation times. Such processes find application, e.g., in the fields of activation rates of chemical reactions and of nonlinear optical systems. Since their statistical dynamics is determined by the temporal evolution of the probability density of the process, the following probabilistic study of the free second-order process (1) and (4) might prove useful in future discussions of nonlinear second-order processes driven by Ornstein-Uhlenbeck noise.

In Sec. II A we derive an exact second-order partial differential equation for the joint density $p(x,v,t)$ and in Sec. II B we obtain the general exact solution for the marginal distribution $p(v,t)$. In particular, the form of $p(v,t)$ for $t\to\infty$ is used as a boundary condition for obtaining asymptotic solutions for $p(x,v,t)$ and for $p(x,t)$ in Sec. III. In Sec. III A we study exact solutions for the distributions $p(x,v,t)$, $p(v,t)$ and $p(x,t)$ in the domain $t \ll \tau$, including the form of the lowest corrections to the limiting expressions for $t\to 0$. In Sec. III B we present the exact forms of the distributions for the limit of a Gaussian white-noise force ($\tau=0$) as a preliminary to the analysis of the long-time distributions in Sec. III C. Our distributions for finite τ in the asymptotic regime, $t \gg \tau$, are not rigorously defined by boundary conditions. Therefore, we compare them at the end of Sec. III C with exact calculations of lower-order moments. In particular, this suggests that our distribution of displacements is actually exact, to the considered order. Some concluding remarks are presented in Sec. IV.

II. FOKKER-PLANCK EQUATIONS FOR PROBABILITY DENSITIES

A. Joint distribution of displacement and velocity

The joint probability density $p(x,v,t)$ for a displacement x and a velocity v at time t is defined by

$$p(x,v,t)=\langle \delta(x-x(t))\delta(v-v(t))\rangle \ , \tag{6}$$

where $x(t)$ and $v(t)$ are the solutions of Eqs. (2a) and (2b) for a given realization $f(t)$:

$$v(t)=\frac{1}{m}\int_0^t dt' f(t') \ , \tag{7a}$$

$$x(t)=\frac{1}{m}\int_0^t dt'\int_0^{t'} dt'' f(t'') \ . \tag{7b}$$

Here we have assumed that the particle was at rest at $t=0$. By taking time derivatives of both sides of (6) we obtain [using the vector notation $\delta(\mathbf{u}-\mathbf{u}(t)) = \delta(x-x(t))\delta(v-v(t))$]

$$\frac{\partial p(x,v,t)}{\partial t} = -\frac{\partial}{\partial x}\left\langle \frac{\partial x(t)}{\partial t}\delta(\mathbf{u}-\mathbf{u}(t))\right\rangle - \frac{\partial}{\partial v}\left\langle \frac{\partial v(t)}{\partial t}\delta(\mathbf{u}-\mathbf{u}(t))\right\rangle \ . \tag{8}$$

By inserting Eqs. (2a) and (2b) in the rhs we get, using the identity $z_0\delta(z-z_0)=z\delta(z-z_0)$,

$$\frac{\partial p(x,v,t)}{\partial t} = -v\frac{\partial p(x,v,t)}{\partial x} - \frac{1}{m}\frac{\partial}{\partial v}\langle f(t)\delta(\mathbf{u}-\mathbf{u}(t))\rangle \ . \tag{9}$$

Since $x(t)$ and $v(t)$ are functionals of the Gaussian random variable $f(t)$, with zero mean value, we may use Novikov's identity [8] for expressing the average in the second term on the rhs in the form

$$\langle f(t)\delta(\mathbf{u}-\mathbf{u}(t))\rangle = \int_0^t dt' \langle f(t)f(t')\rangle \times \left\langle \frac{\delta(\delta(\mathbf{u}-\mathbf{u}(t)))}{\delta f(t')}\right\rangle \ , \tag{10}$$

where the functional derivative $\delta(\delta(\mathbf{u}-\mathbf{u}(t)))/\delta f(t')$ is

$$\frac{\delta(\delta(\mathbf{u}-\mathbf{u}(t)))}{\delta f(t')} = -\frac{\delta x(t)}{\delta f(t')}\frac{\partial\delta(\mathbf{u}-\mathbf{u}(t))}{\partial x} - \frac{\delta v(t)}{\delta f(t')}\frac{\partial \delta(\mathbf{u}-\mathbf{u}(t))}{\partial v} \ , \tag{11}$$

where, from (7a) and (7b),

$$\frac{\delta x(t)}{\delta f(t')} = \frac{1}{m}(t-t'), \quad \frac{\delta v(t)}{\delta f(t')} = \frac{1}{m} \ . \tag{12}$$

Finally, we insert Eqs. (10)–(12) into (9) and, after performing the integrals over t', using (4), we obtain the closed partial differential equation

$$\frac{\partial p(x,v,t)}{\partial t} = \left[-v\frac{\partial}{\partial x} - \alpha b(t)\frac{\partial^2}{\partial x\,\partial v} + \alpha a(t)\frac{\partial^2}{\partial v^2}\right] \times p(x,v,t) \ , \tag{13}$$

where

$$a(t)=1-\exp(-t/\tau) \ , \tag{14a}$$

$$b(t)=(t+\tau)\exp(-t/\tau)-\tau \ , \tag{14b}$$

and $\alpha = f_0^2/2m^2$. For a particle at rest at $t=0$ we have

$$p(x,v,0)=\delta(x)\delta(v) \ . \tag{15}$$

This boundary condition may be used for determining $p(x,v,t)$ in the short-time regime, $t \ll \tau$. On the other hand, for obtaining $p(x,v,t)$ at loing times, such that $t \gg \tau$, we shall use the general form for the marginal distribution of the velocity, $p(v,t)$, which is itself based on (15). Therefore we now discuss the exact form of $p(v,t)$.

B. Marginal distribution of the velocity

The distribution of the velocity, $p(v,t)$, is easily found, either from the moments derived from (7a) or, more directly, from Eq. (13). From (7a) and (4) it follows that the odd moments of $p(v,t)$ vanish, while the Gaussian form of $f(t)$ implies that the even moments are given by

$$\langle v(t)^{2n}\rangle=(2n-1)!![\langle v^2(t)\rangle]^n, \quad n=0,1,2,\ldots . \tag{16}$$

The average of $v^2(t)$ given by the average of (7a) squared is

$$\langle v^2(t)\rangle=f_0^2 m^{-2}\{t-\tau[1-\exp(-t/\tau)]\} . \tag{17}$$

Using (16), the infinite series expression for the characteristic function may be summed in the form

$$\phi(k)=\exp[-k^2\langle v^2(t)\rangle/2] , \tag{18}$$

whose inverse Fourier transform readily yields the Gaussian distribution

$$p(v,t)=[2\pi\langle v^2(t)\rangle]^{-1/2}\exp\{-v^2/[2\langle v^2(t)\rangle]\} . \tag{19}$$

On the other hand, by integrating both sides of (13) over displacements we obtain a generalized diffusion equation for $p(v,t)$,

$$\frac{\partial p(v,t)}{\partial t}=\alpha a(t)\frac{\partial^2 p(v,t)}{\partial v^2} , \tag{20}$$

assuming that $p(x,v,t)$ vanishes at the boundaries $x=\pm\infty$. One readily verifies that this equation is solved by (19).

III. DETAILED SOLUTIONS

For the purpose of obtaining explicit solutions for $p(x,v,t)$ it is convenient to define the double Fourier transform

$$\tilde{p}(\xi,\nu,t)=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dv\, e^{-i\xi x-i\nu v}p(x,v,t) . \tag{21}$$

The Fourier transformation of Eq. (13) then leads to a first-order partial differential equation for $\tilde{p}\equiv\tilde{p}(\xi,\nu,t)$:

$$\frac{\partial\tilde{p}}{\partial t}=\xi\frac{\partial\tilde{p}}{\partial\nu}+\alpha[b(t)\xi\nu-a(t)\nu^2]\tilde{p} , \tag{22}$$

where $\tilde{p}(\xi,\nu,t)$ obeys the boundary condition [Eq. (15)]

$$p(\xi,\nu,0)=1 . \tag{23}$$

A. Short-time domain $t\ll\tau$

The limit $\tau\to\infty$, $f_0^2/(2\tau)\to$ finite constant, F_0^2, of the behavior of $p(x,v,t)$ for $t\ll\tau$ describes the effect of a static random force. In order to study the form of the joint density for $t\ll\tau$ we must retain the effect of successively higher contributions in $1/\tau$ in Eq. (22). For this purpose it is convenient to make the following exact substitutions. First we define a function $\tilde{q}\equiv\tilde{q}(\xi,\nu,t)$ by

$$\tilde{p}=\tilde{q}\exp\left[\frac{\alpha}{\xi}\left[\frac{a(t)}{3}\nu^3-\frac{b(t)}{2}\xi\nu^2\right]\right] , \tag{24}$$

whose substitution in (22) yields

$$\frac{\partial\tilde{q}}{\partial t}=\xi\frac{\partial\tilde{q}}{\partial\nu}+\frac{\alpha}{\xi}\left[\frac{b'(t)}{2}\xi\nu^2-\frac{a'(t)}{3}\nu^3\right]\tilde{q} , \tag{25}$$

where $c'(t)=dc(t)/dt$. From (14a) and (14b) it follows that the coefficient of $\tilde{q}$ on the rhs is proportional to $1/\tau$. Thus, in order to include both the contributions proportional to $1/\tau$ and to $1/\tau^2$ in the parameters which determine $p(x,v,t)$ [namely, the moments $\mu_{m,n}=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dv\, x^m v^n p(x,v,t)$, $m,n=0,1,2,\ldots$] it is expedient to perform the further sequence of successive transformations [$c''(t)=d^2c(t)/dt^2$, $c'''(t)=d^3c(t)/dt^3$]

$$\tilde{q}=\tilde{r}\exp\left[\frac{\alpha}{\xi^2}\left[\frac{a'(t)}{12}\nu^4-\frac{b'(t)}{6}\xi\nu^3\right]\right] , \tag{26a}$$

$$\tilde{r}=\tilde{s}\exp\left[\frac{\alpha}{\xi^3}\left[\frac{a''(t)}{60}\nu^5-\frac{b''(t)}{24}\xi\nu^4\right]\right] , \tag{26b}$$

$$\tilde{s}=\tilde{t}\exp\left[-\frac{\alpha b'''(t)}{120}\frac{\nu^5}{\xi^3}\right] , \tag{26c}$$

from which it follows that, up to terms proportional to τ^{-3}, $\tilde{t}\equiv\tilde{t}(\xi,\nu,t)$ obeys

$$\frac{\partial\tilde{t}}{\partial t}=\xi\frac{\partial\tilde{t}}{\partial\nu} , \tag{27}$$

whose general solution is an arbitrary function of $t+\nu/\xi$:

$$\tilde{t}(\xi,\nu,t)=\phi\left[t+\frac{\nu}{\xi}\right] . \tag{28}$$

This function is determined from (24) and (26a)–(26c) by imposing the boundary condition (23) which yields

$$\phi(z)=\exp\left[-\frac{\alpha\xi^2}{8\tau}z^4\left[1-\frac{4}{15\tau}z\right]\right] . \tag{29}$$

Clearly the solution for $\tilde{p}(\xi,\nu,t)$ defined by (24), (26a)–(26c), (28), and (29) includes exactly the effect of the random force at orders τ^{-1} and τ^{-2} for the short-time domain. Finally, by expanding $a(t)$, and $b(t)$ in (14a) and (14b) and their derivatives to second order in powers of t/τ we obtain, after some cancellations, the following exponentially bilinear expression for $\tilde{p}(\xi,\nu,t)$:

$$\tilde{p}(\xi,\nu,t)=\exp\left\{-\frac{\alpha t^2}{2\tau}\left[\frac{t^2}{4}\left[1-\frac{4t}{15\tau}\right]\xi^2+\left[1-\frac{t}{3\tau}\right](t\xi\nu+\nu^2)\right]\right\} , \tag{30}$$

where the exponential exponent is exact to order t^2/τ^2. The marginal distribution of the velocity obtained from (30) is

$$\begin{aligned}p(v,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu\, e^{iv\nu}\tilde{p}(0,\nu,t)\\&=\left[2\pi\frac{\alpha t^2}{\tau}\left[1-\frac{t}{3\tau}\right]\right]^{-1/2}\\&\quad\times\exp\left[-\frac{\tau v^2}{2\alpha t^2}\left[1-\frac{t}{3\tau}\right]^{-1}\right] ,\end{aligned} \tag{31}$$

which agrees exactly with (19) when $\langle v^2(t)\rangle$ in (17) is approximated by the first two terms of its infinite series expression in powers of t/τ.

Similarly, the marginal distribution of the displacement is given by the Gaussian expression

$$p(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi\, e^{ix\xi}\tilde{p}(\xi,0,t)$$
$$=\frac{1}{[2\pi\langle x^2(t)\rangle]^{1/2}}\exp\left[-\frac{x^2}{2\langle x^2(t)\rangle}\right], \quad (32)$$

where the mean squared displacement

$$\langle x^2(t)\rangle=\frac{f_0^2}{8m^2\tau}\left[t^4-\frac{4t^5}{15\tau}\right] \quad (33)$$

is exact to order t/τ. The leading term $\langle x^2(t)\rangle\sim t^4$ describes a uniformly accelerated spreading of the distribution of the displacement due to a static Gaussian force (which corresponds to the limits $\tau\to\infty$, $f_0^2/(2\tau)\to F_0^2>0$). On the other hand, the term proportional to t^5 describes the lowest-order reduction of $\langle x^2(t)\rangle$, due to the lowering of the velocity as a result of the increased disorder when τ is noninfinite. Clearly, terms of higher order in $1/\tau$ [i.e., terms proportional to $\tau^{-3},\tau^{-4},\ldots$ in the exponent of (30)] may generally lead to deviations from the Gaussian behavior (32).

The expressions for the distributions of the displacement and of the velocity in the static case readily follow from a simple argument. For a Gaussian force with a static autocorrelation, $\langle f(t)f(t')\rangle=F_0^2$, the moments of the probability density for any frequency-Fourier component [4] of $f(t)$, except the uniform component, are zero. This implies that $f(t)$ is constant, $f(t)=f$, with a Gaussian distribution,

$$p_f(f)=(2\pi F_0^2)^{-1/2}\exp(-f^2/2F_0^2) . \quad (34)$$

Now, for a static force the solutions (7a) and (7b) are $v(t)=ft/m$ and $x(t)=ft^2/(2m)$, so that, from the definitions $p(x,t)=\int_{-\infty}^{\infty}df p_f(f)\delta(x-x(t))$ and $p(v,t)=\int_{-\infty}^{\infty}df p_f(f)\delta(v-v(t))$, we have $p(x,t)=p_f[(2m/t^2)x]$ and $p(v,t)=p_f[(m/t)v]$. These expressions coincide with (32) and (31), respectively, when $\langle x^2(t)\rangle$ and $\langle v^2(t)\rangle$ are approximated by the leading terms, $\langle x^2(t)\rangle=f_0^2t^4(8m^2\tau)^{-1}$ and $\langle v^2(t)\rangle=f_0^2t^2(2m^2\tau)^{-1}$, of (33) and (17) for $t\ll\tau$.

Finally, one may inquire about the form of the joint density of the displacement and the velocity. By double Fourier inversion of (30) we obtain

$$p(x,v,t)=\frac{\tau}{\pi\alpha t^3}\left[\frac{15\tau}{t}\left[1-\frac{t}{3\tau}\right]^{-1}\right]^{1/2}$$
$$\times\exp\left[-\frac{30\tau^2}{\alpha t^5}\left[x-\frac{vt}{2}\right]^2-\frac{\tau}{2\alpha t^2}\left[1-\frac{t}{3\tau}\right]^{-1}v^2\right] . \quad (35)$$

B. Gaussian white noise ($\tau=0$)

The white-noise limit is useful because it describes the limiting asymptotic form of the distributions for time intervals large compared to the correlation time. For $\tau=0$ Eq. (22) reduces to

$$\frac{\partial\tilde{p}}{\partial t}=\xi\frac{\partial\tilde{p}}{\partial\nu}-\alpha\nu^2\tilde{p} , \quad (36)$$

whose exact solution is

$$\tilde{p}(\xi,\nu,t)=\phi\left[t+\frac{\nu}{\xi}\right]\exp\left[\frac{\alpha\nu^3}{3\xi}\right] . \quad (37)$$

From the boundary condition (23) we get $\phi(z)=\exp[-(\alpha\xi^2/3)z^3]$, which leads to

$$\tilde{p}(\xi,\nu,t)=\exp\left[-\alpha t\left[\frac{t^2\xi^2}{3}+t\xi\nu+\nu^2\right]\right] . \quad (38)$$

The Fourier inversion of $\tilde{p}(0,\nu,t)$ and of $\tilde{p}(\xi,0,t)$ yields successively the densities of v and of x:

$$p(v,t)=\left[\frac{2\pi f_0^2t}{m^2}\right]^{-1/2}\exp\left[-\frac{mv^2}{2f_0^2t}\right], \quad (39)$$

$$p(x,t)=\left[\frac{2\pi f_0^2t^3}{3m^2}\right]^{-1/2}\exp\left[-\frac{3m^2x^2}{2f_0^2t^3}\right], \quad (40)$$

with mean-squared deviations

$$\langle v^2(t)\rangle=\frac{f_0^2t}{m^2} , \quad (41)$$

$$\langle x^2(t)\rangle=\frac{f_0^2t^3}{3m^2} . \quad (42)$$

Here $\langle x^2(t)\rangle$ describes the anomalous, so-called superdiffusive spreading due to the time-dependent fluctuating force. In fact, the distribution (40) verifies a diffusion equation with a time-dependent diffusion coefficient derived by Masoliver [1] for Gaussian white noise. For the sake of completeness we also give the joint distribution $p(x,v,t)$ which follows from (38), namely

$$p(x,v,t)=\frac{\sqrt{3}}{2\pi\alpha t^2}\exp\left[-\frac{1}{\alpha t}\left[\frac{3x^2}{t^2}-\frac{3xv}{t}+v^2\right]\right] . \quad (43)$$

C. Long-time domain $t\gg\tau$

In order to study the solution of (22) in the asymptotic regime $t\gg\tau$ we first eliminate the second term on the rhs by means of the transformation

$$\tilde{p}(\xi,\nu,t)=\tilde{q}(\xi,\nu,t)\times\exp\left[-\alpha\int_0^t dt'[a(t')\nu^2-b(t')\xi\nu]\right] , \quad (44)$$

where $\tilde{q}\equiv\tilde{q}(\xi,\nu,t)$ obeys

$$\frac{\partial\tilde{q}}{\partial t}=\xi\frac{\partial\tilde{q}}{\partial\nu}-\alpha\xi\int_0^t[2a(t')\nu-b(t')\xi]dt'\tilde{q} . \quad (45)$$

Since now the exponential terms appearing in $\int_0^t dt' a(t') = t - \tau + \tau \exp(-t/\tau)$ and in $\int_0^t dt' b(t') = -\tau t + 2\tau^2 - \tau(t+2\tau)\exp(-1/\tau)$ are proportional to τ they may be dropped if one restricts to contributions in the parameters of $\tilde{p}(\xi,\nu,t)$ which are independent of or linear in τ for $t \gg \tau$. Indeed, the terms of the form $\tau\exp(-t/\tau)$ in (45) lead to contributions proportional to higher powers, $\tau^n (n>1)$, in $\tilde{q}(\xi,\nu,t)$. In the above approximation (45) reduces to

$$\frac{\partial \tilde{q}}{\partial t} = \xi \frac{\partial \tilde{q}}{\partial \nu} - 2\alpha(t-\tau)\xi\nu\tilde{q} - \alpha t\tau\xi^2\tilde{q} \ , \tag{46}$$

whose exact solution, obtained by a method similar to that of Sec. III A, is

$$\tilde{q}(\xi,\nu,t) = \phi\left[t + \frac{\nu}{\xi}\right]\exp\left\{\frac{\alpha}{\xi}\left[\left[t - \frac{\tau}{2}\right]\xi\nu^2 + t\tau\xi^2\nu + \frac{\nu^3}{3}\right]\right\}, \tag{47}$$

where $\phi(z)$ is an arbitrary function. We now determine $\phi(z)$ using the exact solution (17) and (19) for the marginal distribution of the velocity as a boundary condition. By Fourier transforming (19) we get

$$\tilde{p}(0,\nu,t) = \exp[-\alpha(t-\tau)\nu^2], \quad t \gg \tau \ , \tag{48}$$

where we have omitted the term $2\alpha\tau\exp(-t/\tau)$ in the exponent since we restrict to contributions linear in τ for $t \gg \tau$, as discussed above. From (44), (47), and (48) we then obtain

$$\lim_{\xi\to 0}\phi\left[t + \frac{\nu}{\xi}\right] = \exp\left\{-\frac{\alpha\xi^2}{3}\left[\frac{\nu^3}{\xi^3} + 3\left[t - \frac{\tau}{2}\right]\frac{\nu^2}{\xi^3}\right]\right\}. \tag{49}$$

The explicit form for the function $\phi(t+\nu/\xi)$ for finite ξ which has the limit (49) is

$$\phi\left[t + \frac{\nu}{\xi}\right] = \exp\left[-\frac{\alpha\xi^2}{3}\left[t + \frac{\nu}{\xi}\right]^3 + \frac{\alpha\tau\xi^2}{2}\left[t + \frac{\nu}{\xi}\right]^2\right]. \tag{50}$$

Indeed the exponent in (49) is given by the first two leading terms of the exponent in (50) for $\xi \to 0$, at finite ν. Expression (50) also reduces to the exact form obtained in Sec. III B for Gaussian white noise. We note, however, that the above determination of $\phi(z)$, based on the boundary condition for $p(v,t)$, is generally approximate since the exponent on the rhs of (49) is compatible with the dependence of ϕ on $t + (\nu/\xi)$ only up to terms proportional to ν/ξ and terms independent of ν/ξ.

With expression (50) we obtain

$$\tilde{p}(\xi,\nu,t) = \exp\left[-\frac{\alpha t^3}{3}\left[1 - \frac{3\tau}{2t}\right]\xi^2 - \alpha t^2\left[1 - \frac{\tau}{t}\right]\xi\nu - \alpha t\left[1 - \frac{\tau}{t}\right]\nu^2\right]. \tag{51}$$

The Fourier inversion of $\tilde{p}(\xi,0,t)$ then yields a Gaussian distribution for the displacement

$$p(x,t) = \frac{1}{[2\pi\langle x^2(t)\rangle]^{1/2}}\exp\left[-\frac{x^2}{2\langle x^2(t)\rangle}\right], \tag{52}$$

with a mean-squared displacement

$$\langle x^2(t)\rangle = \frac{f_0^2}{m^2}\left[\frac{t^3}{3} - \frac{\tau t^2}{3}\right], \quad t \gg \tau \ . \tag{53}$$

Equation (53) coincides with (42) for $\tau = 0$ and with the two leading terms of the asymptotic limit of the exact mean-squared displacement for finite τ [1,6]. On the other hand, the inverse transform of (51) is

$$p(x,v,t) = \frac{\sqrt{3}}{2\pi\alpha t[(t-\tau)(t-3\tau)]^{1/2}} \times \exp\left[-\frac{v^2}{4\alpha(t-\tau)} - \frac{3(x-vt)^2}{\alpha t^2(t-3\tau)}\right]. \tag{54}$$

Finally, we comment on the accuracy of our form (50) for $\phi[t + (\nu/\xi)]$. To this end we study the moments, $\mu_{m,n}(t)$, of the bivariate distribution $p(x,v,t)$,

$$\mu_{m,n} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\, x^m v^n p(x,v,t) \ , \quad m,n = 0,1,2,\ldots \ , \tag{55}$$

where, in particular, $\mu_{m,0}$ and $\mu_{0,m}$ are the moments of the distributions of the displacement $p(x,t)$, and of the velocity, $p(v,t)$, respectively. By multiplying both sides of (13) by $x^m v^n$ and integrating by parts on the rhs we get

$$\frac{d\mu_{m,n}}{dt} = m\mu_{m-1,n+1} - mn\alpha b(t)\mu_{m-1,n-1} + n(n-1)\alpha a(t)\mu_{m,n-2}, \quad m,n = 0,1,2,\ldots \ . \tag{56}$$

The boundary terms appearing on integration by parts vanish because of the condition of normalization and of bounded moments for any given t. The boundary conditions for (56), which follow from (15), are

$$\mu_{0,m}(0) = \mu_{m,0}(0) = \delta_{m,0}, \mu_{m,n}(0) = \delta_{m,0}\delta_{n,0} \ . \tag{57}$$

As shown above, expression (50) and the resulting form (51) for $\tilde{p}(\xi,\nu,t)$ yield the correct asymptotic form for $\mu_{2,0}(t)$ to order τ/t. Thus, in order to check the accuracy of the Gaussian distribution (52) we now compare its higher moments,

$$\mu_{2m,0}^{\text{Gaussian}}(t) = (2m-1)!![\mu_{2,0}(t)]^m \ , \tag{58}$$

with the exact moments obtained from (56) to linear order in τ/t for $t \gg \tau$. In solving (56) to order τ/t we approximate (14a) and (14b) by $a(t) = 1$ and $b(t) = -\tau$ and we use Eqs. (16 and 17) for the velocity moments in the form $\mu_{0,2n}(t) = (2n-1)!!(2\alpha t)^n(1 - n\tau/t)$, $\mu_{0,2n+1}(t) = 0$, as well as the form (53) for $\mu_{2,0}(t)$. Indeed, it turns out that in the process of solving (56) for successively higher moments in terms of the lowest ones, the exponentials in $a(t)$ and $b(t)$ enter only in combination with powers of t

and/or τ, thus giving rise (via partial integrations) to higher-order terms in τ only. The calculation of $\mu_{4,0}(t)$ and $\mu_{6,0}(t)$ based on these results and on the following determination of mixed moments in the coupled equations (56) ($\epsilon=\tau/t$):

$$\mu_{1,1}=\alpha t^2(1-\epsilon) ,$$
$$\mu_{1,3}=6\alpha^2 t^3(1-2\epsilon) ,$$
$$\mu_{1,5}=60\alpha^3 t^4(1-3\epsilon) ,$$
$$\mu_{2,2}=\tfrac{10}{3}\alpha^2 t^4(1-\tfrac{11}{5}\epsilon) ,$$
$$\mu_{2,4}=32\alpha^3 t^5(1-\tfrac{25}{8}\epsilon) ,$$
$$\mu_{3,3}=18\alpha^3 t^6(1-\tfrac{10}{3}\epsilon) ,$$
$$\mu_{3,1}=2\alpha^2 t^5(1-\tfrac{5}{2}\epsilon) ,$$
$$\mu_{4,2}=\tfrac{32}{3}\alpha^3 t^7(1-\tfrac{29}{8}\epsilon) ,$$
$$\mu_{5,1}=\tfrac{20}{3}\alpha^3 t^8(1-4\epsilon) ,$$

yields the following expressions, which are exact to order τ/t:

$$\mu_{4,0}(t)=\frac{4\alpha^2 t^6}{3}\left[1-3\frac{\tau}{t}\right] , \tag{59}$$

$$\mu_{6,0}(t)=\frac{40\alpha^3 t^9}{9}\left[1-\frac{9}{2}\frac{\tau}{t}\right] . \tag{60}$$

We have also verified explicitly that $\mu_{1,0}=0$, $\mu_{3,0}=0$, $\mu_{5,0}=0,\ldots$, which clearly implies that $p(x,t)$ is symmetric, $p(x,t)=p(-x,t)$. Expressions (59 and 60) coincide with the corresponding Gaussian moments (58) at the considered order. This strongly suggests that the Gaussian distribution (52) is actually exact to order τ/t.

IV. CONCLUDING REMARKS

In this paper we have discussed the joint density and the corresponding marginal densities for the displacement and the velocity of a free particle subjected to a Gaussian applied force $f(t)$, which is exponentially correlated. We have derived an exact Fokker-Planck equation for the joint distribution, whose solution we have studied analytically in the limits of time intervals short and long compared to the correlation time τ. In both cases we have included the effect of the lowest correction to the asymptotic behavior of the parameters (moments) which determine the distributions. Exact expressions for arbitrary times have also been found for the marginal distribution of velocities for any τ, as well as for the joint distribution of displacements and velocities and for the corresponding marginal distributions in the case of Gaussian white noise ($\tau=0$).

We conclude with a brief comparison between our results for exponentially correlated Gaussian noise and those obtained recently by Masoliver [1] for the same system with exponentially correlated dichotomous noise. In the latter case the joint density obeys a third-order partial differential equation instead of our Fokker-Planck equation (13) with variable coefficients. The marginal distribution of the velocity, on the other hand, obeys the telegrapher's equation in the dichotomous case [1] and the diffusion equation (20) with a variable diffusion coefficient in the Gaussian case. However, for $t\to\infty$ the distribution of velocities in the dichotomous case reduces to the Gaussian distribution (19) with the limiting form for $t\to\infty$ for the mean-square deviation, $\langle v^2(t)\rangle\sim t$. Finally, for exponentially correlated dichotomous noise the distribution of displacements obeys a telegrapher's equation with variable coefficients [1], whose solution for $t\to\infty$ follows the same diffusion process, with a variance $\langle x^2(t)\rangle\sim t^3$, as found above for the Gaussian case [Eqs. (52 and 53) for $t\to\infty$]. We also note that in the Gaussian white-noise limit Masoliver's third-order partial differential equation for $p(x,v,t)$ reduces to a Fokker-Planck equation [his equation (3.27)] which is identical to Eq. (13) for $\tau=0$ [$b(t)=0$, $a(t)=1$]. In this case our results for $p(x,t)$ and $p(v,t)$ in Sec. III B coincide with those given by Masoliver [1].

Finally, it would be interesting to compare the results of Sec. III A for the short-time regime with corresponding results for exponentially correlated dichotomous noise. Unfortunately, Masoliver has not discussed explicit solutions for the short-time domain. He also did not discuss solutions analogous to (54) and (43) for the joint density, $p(x,v,t)$, in the long-time ($t\gg\tau$) and in the Gaussian white-noise ($\tau=0$) regimes.

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